Chapter 12

Semi-Markov Processes

12.1. Positive (J-X) processes

Let us consider a *physical* or *economic system* called S with m possible states, m being a finite natural number.

For simplicity, we will note by I the set of all possible states:

$$I = \{1, ..., m\} \tag{12.1}$$

as we did in Chapter 11.

At time 0, system S starts from an initial state represented by the r.v. J_0, stays a non-negative random length of time X_1 in this state, and then goes into another state J_1 for a non-negative length of time X_2 before going into J_2, etc.

So we have a two-dimensional stochastic process in discrete time called a *positive* (J-X) *process*:

$$(J - X) = ((J_n, X_n), n \geq 0) \tag{12.2}$$

assuming

$$X_0 = 0, \ a.s. \tag{12.3}$$

where the sequence $(J_n, n \geq 0)$ gives the successive *states* of S in time and the sequence $(X_n, n \geq 0)$ gives the successive *sojourn times*.

More precisely, X_n is the time spent by S in state J_{n-1} ($n>0$).

Times at which transitions occur are given by the sequence $(T_n, n \geq 0)$ where:

$$T_0 = 0,\ T_1 = X_1, \ldots, T_n = \sum_{r=1}^{n} X_r \tag{12.4}$$

and so

$$X_n = T_n - T_{n-1},\ n \geq 1 . \tag{12.5}$$

12.2. Semi-Markov and extended semi-Markov chains

On the complete probability space $(\Omega, \Im, P)$, the stochastic dynamic evolution of the considered (J-X) process will be determined by the following assumptions:

$$P(X_0=0)=1,\ \text{a.s.},$$
$$P(J_0=i)=p_i,\ i=1,\ldots,m \text{ with } \sum_{i=1}^{m} p_i = 1, \tag{12.6}$$

for all $n>0$, $j=1,\ldots,m$, we have:

$$P(J_n = j, X_n \leq x \,|\, (J_k, X_k), k = 0,\ldots,n-1) = Q_{J_{n-1}j}(x),\ a.s. \tag{12.7}$$

where any function Q_{ij} (i,j=1,…,m) is a non-decreasing real function null on $\mathbb{R}^+$ such that if

$$p_{ij} = \lim_{x \to +\infty} Q_{ij}(x),\ i, j \in I , \tag{12.8}$$

then:

$$\sum_{j=1}^{m} p_{ij} = 1,\ i \in I . \tag{12.9}$$

With matrix notation, we will write:

$$\mathbf{Q} = \left[Q_{ij} \right],\ \mathbf{P} = \left[p_{ij} \right] (= \mathbf{Q}(\infty)),\ \mathbf{p} = (p_1, \ldots, p_m) . \tag{12.10}$$

This leads to the following definitions.

Definition 12.1 *Every matrix* $m \times m\, Q$ *of non-decreasing functions null on* $\mathbb{R}^+$ *satisfying properties (12.8) and (12.9) is called a semi-Markov matrix or a semi-Markov kernel.*

Definition 12.2 Every couple (p,Q) where Q is a semi-Markov kernel and *p* a vector of initial probabilities defines a positive (*J*,*X*) process

$$(J,X) = ((J_n,X_n), n \geq 0) \text{ with } I \times \mathbb{R}^+$$

as state space, also called a semi-Markov chain (*SMC*).

Sometimes, it is useful that the random variables $X_n, n \geq 0$ take their values in $\mathbb{R}$ instead of $\mathbb{R}^+$, in which case, we need the next two definitions.

Definition 12.3 *Every matrix* $m \times m$ Q *of non-decreasing functions satisfying properties* (12.8) *and* (12.9) *is called an extended semi-Markov matrix or an extended semi-Markov kernel.*

Definition 12.4 *Every couple* (**p,Q**) *where* **Q** *is an extended semi-Markov kernel and* **p** *a vector of initial probabilities defines a* (*J*,*X*) *process* (*J*,*X*) = ((J_n,X_n), $n \geq 0$) *with* $I \times \mathbb{R}$ *as state space, also called an extended semi-Markov chain (ESMC).*

Let us return to the main condition (12.7); its meaning is clear. For example if we assume that we observe for a certain fixed *n* that $J_{n-1} = i$, then the basic relation (12.7) gives us the value of the following conditional probability:

$$P(J_n = j, X_n \leq x \,|\, (J_k, X_k), k = 0,...,n-1, J_{n-1} = i) = Q_{ij}(x). \tag{12.11}$$

That is, the knowledge of the value of J_{n-1} suffices to give the conditional probabilistic evolution of the future of the process whatever the values the other past variables might be.

According to Kingman (1972), the event $\{\omega : J_{n-1}(\omega) = i\}$ is *regenerative* in the sense that the observation of this event gives the complete evolution of the process in the future as it could evolve from *n* = 0 with *i* as the initial state.

(*J*-*X*) processes will be fully developed in section 12.4.

Remark 12.1 The second member of the semi-Markov characterization property (12.7) does not explicitly depend on *n*; also we can be precise that we are now studying *homogenous* semi-Markov chains in opposition with the *non-homogenous* case where this dependence with respect to *n* is valid.

12.3. Primary properties

We will start by studying the marginal stochastic processes $(J_n, n \geq 0)$, $(X_n, n \geq 0)$ called the *J-process* and the *X-process* respectively.

The J-process

From properties of the conditional expectation, the process $(J_n, n \geq 0)$ satisfies the following property:

$$P(J_n = j \,|\, (J_k, X_k), k = 0,...,n-1) = Q_{J_{n-1}j}(+\infty)\,. \tag{12.12}$$

Using the smoothing property (see property (10.150)) of conditional expectation, we obtain

$$P(J_n = j \,|\, (J_k), k = 0,...,n-1) = E(Q_{J_{n-1}j}(+\infty) \,|\, (J_k), k = 0,...,n-1), \tag{12.13}$$

and as the r.v. $Q_{J_{n-1}j}(+\infty)$ is $(J_k, k = 0,...n-1)$,k=0,…,n-1)-measurable, we finally obtain from relation (12.8) that:

$$P(J_n = j \,|\, (J_k), k = 0,...,n-1) = p_{J_{n-1}j}\,. \tag{12.14}$$

Since relation (12.9) implies that matrix **P** is a Markov matrix, we have thus proved the following result.

Proposition 12.1 *The J-process is a homogenous Markov chain with P as its transition matrix.*

That is the reason why this *J*-process is called the *embedded Markov chain* of the considered SMC in which the r.v. J_n represents the state of the system S just after the nth transition.

From results of Corollary 11.1, it follows that in the ergodic case there exists one and only one stationary distribution of probability $\boldsymbol{\pi} = (\pi_1, ..., \pi_m)$ satisfying:

$$\begin{aligned} &\pi_i = \sum_{j=1}^{m} \pi_j p_{ji},\ j = 1,...,m, \\ &\sum_{i=1}^{m} \pi_i = 1 \end{aligned} \tag{12.15}$$

such that

$$\lim_{n\to\infty} P(J_n = j \mid J_0 = i)(= \lim_{n\to\infty} p_{ij}^{(n)}) = \pi_j, i, j \in I, \tag{12.16}$$

where we know from relation (11.22) that

$$\left[p_{ij}^{(n)} \right] = \mathbf{P}^n . \tag{12.17}$$

The X-process

Here, the situation is entirely different for the fact that the distribution of X_n depends on $J_{n\text{-}1}$. Nevertheless, we have an interesting property of *conditional independence*, but before giving this property we must introduce some definitions.

Definition 12.5 *The two following conditional probability distributions:*

$$\begin{aligned} &F_{J_{n-1}J_n}(x) = P(X_n \le x \mid J_{n-1}, J_n), \\ &H_{J_{n-1}}(x) = P(X_n \le x \mid J_{n-1}) \end{aligned} \tag{12.18}$$

are respectively called the conditional and unconditional distributions of the sojourn time X_n.

From the general properties of conditioning recalled in section 10.2, we successively obtain

$$\begin{aligned} F_{J_{n-1}J_n}(x) &= E\left(P(X_n \le x \mid (J_k, X_k), k \le n-1, J_n) \mid J_{n-1}, J_n \right), \\ &= E\left(\frac{Q_{J_{n\text{-}1}J_n}(x)}{p_{J_{n\text{-}1}J_n}} \middle| J_{n-1}, J_n \right), \\ &= \frac{Q_{J_{n\text{-}1}J_n}(x)}{p_{J_{n\text{-}1}J_n}}, \end{aligned} \tag{12.19}$$

provided that $p_{J_{n-1}J_n}$ is strictly positive. If not, we can arbitrarily give to (12.19) for example the value $U_1(x)$ defined as

$$U_1(x) = \begin{cases} 0, x < 0, \\ 1, x \ge 0. \end{cases} \tag{12.20}$$

Moreover, from the smoothing property, we also have:

$$H_{J_{n-1}}(x)(=P(X_n \leq x \mid J_{n-1})) = E(F_{J_{n-1}J_n}(x) \mid J_{n-1}),$$
$$= \sum_{j=1}^{m} p_{J_{n-1}J_n} F_{J_{n-1}J_n}(x). \tag{12.21}$$

We have thus proved the following proposition.

Proposition 12.2 *As a function of the semi-kernel* **Q**, *the conditional and unconditional distributions of the sojourn time X_n are given by:*

$$F_{ij}(x)(=P(X_n \leq x \mid J_{n-1}=i, J_n = j)) = \begin{cases} \dfrac{Q_{ij}(x)}{p_{ij}}, p_{ij} > 0, \\ U_1(x), p_{ij} = 0, \end{cases} \tag{12.22}$$
$$H_i(x)(=P(X_n \leq x \mid J_{n-1} = i)) = \sum_{j=1}^{m} p_{ij} F_{ij}(x).$$

Remark 12.2

(a) From relation (12.22), we can also express kernel **Q** as a function of F_{ij}, $i,j=1,\ldots,m$:

$$Q_{i\,j}(x) = p_{ij} F_{ij}(x), i, j \in I, x \in \mathbb{R}^+. \tag{12.23}$$

So, every SMC can also be characterized by the triple (**p**,**P**,**F**) instead of the couple (**p**,**Q**) where the $m \times m$ matrix **F** is defined as $\mathbf{F} = \left[F_{ij} \right]$, and where the functions $F_{ij}, i, j = 1, \ldots, m$ are distribution functions on support $\mathbb{R}^+$.

(b) We can also introduce the *means* related to these conditional and unconditional distribution functions.

When they exist we will note:

$$\beta_{ij} = \int_R x dF_{i\,j}(x), i, j = 1, \ldots, m,$$
$$\eta_i = \int_R x dH_i(x), i = 1, \ldots, m \tag{12.24}$$

and relation (12.22) leads to the relation:

$$\eta_i = \sum_{j=1}^{m} p_{ij} \beta_{ij} \ . \tag{12.25}$$

The quantities β_{ij}, $i,j = 1,\ldots,m$ and η_i, $I = 1,\ldots,m$ are respectively called the *conditional* and *unconditional means* of the sojourn times.

We can now give the property of *conditional independence*.

Proposition 12.3 *For each integer k, if $n_1,n_2,\ldots,n_k$ are k positive integers such that $n_1<n_2<\cdots<n_k$ and $x_{n_1},\ldots,x_{n_k}$ k are real numbers. We have:*

$$\begin{aligned} &P\left(X_{n_1} \le x_{n_1},\ldots,X_{n_k} \le x_{n_k} \,\middle|\, J_{n_1-1}, J_{n_1},\ldots,J_{n_k-1}, J_{n_k}\right) \\ &= F_{J_{n_1-1}J_{n_1}}(x_{n_1})\ldots F_{J_{n_k-1}J_{n_k}}(x_{n_k}), \end{aligned} \tag{12.26}$$

that is, k random variables $X_{n_1},\ldots,X_{n_k}$ are conditionally independent given $J_{n_1-1}, J_{n_1},\ldots,J_{n_k-1}, J_{n_k}$.

The T-process

By relation (12.4), the sequence $(T_n, n \ge 0)$ represents successive *renewal epochs*, that is, times at which transitions occur.

By analogy with renewal theory, we have the following definition.

Definition 12.6 *The two-dimensional process $((J_n,T_n), n \ge 0)$ is called the Markov renewal process of kernel* **Q**.

Before giving the expression of the marginal distribution of the random vector (J_n,T_n) with values in $I \times \mathbb{R}^+$, given that $J_0=i$, let us define the *marginal distribution* of the (J,T) process $((J_n,T_n), n \ge 0)$:

$$Q_{ij}^n(t) = P(J_n = j, T_n \le t \,|\, J_0 = i),\ i,j \in I,\ n \ge 0,\ t \ge 0\ . \tag{12.27}$$

With $\mathbf{A} = \left[A_{ij}\right]$ and $\mathbf{B} = \left[B_{ij}\right]$, two $m \times m$ matrices of integrable functions, we associate a new matrix $\mathbf{A} \bullet \mathbf{B}$ whose general element $(\mathbf{A} \bullet \mathbf{B})_{ij}$ is the function of t defined by:

$$(\mathbf{A} \bullet \mathbf{B})_{ij}(t) = \sum_{k=1}^{m} \int_{\mathbb{R}} A_{kj}(t-y)\, dB_{ik}(y). \tag{12.28}$$

It can be easily seen that this type of product, called the *convolution product for matrices*, is *associative* but not always commutative.

In the particular case of A=B, we set:

$$\begin{aligned}&\mathbf{A}\bullet\mathbf{A}=\mathbf{A}^{(2)},\ldots,\mathbf{A}\bullet\cdots\bullet\mathbf{A}=\mathbf{A}^{(n)}\left(=\left[A_{ij}^{(n)}\right]\right),\\&\mathbf{A}^{(0)}=(\delta_{ij}U_0),\mathbf{A}^{(1)}=\mathbf{A}.\end{aligned} \tag{12.29}$$

If all the functions $A_{ij}, B_{ij}, i, j=1,\ldots,m,$ vanish at $-\infty$, we can also use an integration by parts to express (12.28) as follows:

$$(\mathbf{A}\bullet\mathbf{B})_{ij}(t)=\sum_{k=1}^{m}\int_{\mathbb{R}} B_{ik}(t-y)dA_{kj}(y) \tag{12.30}$$

and moreover if **A**=**B**, we obtain:

$$(\mathbf{A}\bullet\mathbf{B})_{ij}(t)=\sum_{k=1}^{m}\int_{\mathbb{R}} A_{ik}(t-y)dA_{kj}(y)\ . \tag{12.31}$$

Proposition 12.4 *For all* $n\geq 0$*, we have:*

$$Q_{ij}^{n}=Q_{ij}^{(n)}\ . \tag{12.32}$$

Moreover, we also have:

$$\lim_{t\to\infty}Q^{(n)}(t)=P^{n}\ . \tag{12.33}$$

12.4. Examples

Semi-Markov theory is one of the most productive subjects of stochastic processes to generate applications in real-life problems, particularly in the following fields: economics, manpower models, insurance, finance (more recently), reliability, simulation, queuing, branching processes, medicine (including survival data), social sciences, language modeling, seismic risk analysis, biology, computer science, chromatography and fluid mechanics.

Important results in such fields may be found in Janssen (1986), Janssen and Limnios (1999), and Janssen and Manca (2006 and 2007).

Let us give three examples in the fields of insurance and reliability.

Example 12.1 *The claim process in insurance*

Let us consider an insurance company covering m types of risks or having m different types of customers for the same risk forming the set $I=\{1,\dots,m\}$.

For example, in automobile insurance, we can distinguish three types of drivers: *good, average* and *bad* and so I is a space consisting of three states: 1 for good, 2 for average and 3 for bad.

Now, let $(X_n, n \geq 1)$ represent the sequence of successive observed *claim amounts,* $(Y_n, n \geq 1)$ the sequence of interarrivals between two successive claims and $(J_n, n \geq 1)$ successive *types of observed risks.*

In the traditional model of risk theory called the Cramer Lundberg model (1909, 1955), it is assumed with that there is only one type of risk and the claim arrival process is a *Poisson process* parameter λ; later, Andersen (1967) extends this model to an arbitrary *renewal process* and moreover in these two traditional models, the process of claim amounts is a renewal process independent of the claim arrival process.

The consideration of an SMC for the two-dimensional processes $((J_n, X_n), n \geq 0)$ and/or $((J_n, Y_n), n \geq 0)$ provides the possibility to introduce a certain dependence between the successive claim amounts. This model was first developed by Janssen (1969b, 1970, 1977) along the lines of Miller's work (1962) and since then has led to many extensions; see for example Asmussen (2000).

Example 12.2 *Occupational illness insurance*

This problem is related to occupational illness insurance with the possibility of leading to partial or permanent disability. In this case, the amount of the incapacitation allowance depends on the degree of disability recognized in the policyholder by the occupational health doctor, in general on an annual basis, because this degree is a function of an occupational illness which can take its course.

Considering as in the example in section 11.6.2 this invalidity degree as a stochastic process $(J_n,\ n \geq 0)$ where J_n represents the value of this degree when the illness really takes its course, and we must then introduce the r.v. X_n representing the time between two successive transitions from J_{n-1} to J_n.

In practice, these transitions can be observed with periodic medical inspections.

The assumption that the *J*-*X* process is an SMC extends the Markov model of Chapter 11 and is fully discussed in Janssen and Manca (2006).

Example 12.3 *Reliability*

There are many semi-Markov models in reliability theory; see for example Osaki (1985) and more recently Limnios and Oprisan (2001), (2003).

Let us consider a *reliability system* S that can be at any time t in one of the m states of $I=\{1,\ldots,m\}$.

The stochastic process of the successive states of S is represented by $S=\left(S_t, t\geq 0\right)$.

The state space I is partitioned into two sets U and D so that

$$I=U\bigcup D,\ U\bigcap D=\varnothing,\ U\neq\varnothing,\ D\neq\varnothing. \tag{12.34}$$

The interpretation of these two sets is the following: the subset U contains all "good" states, in which the system is working and the subset D of all "bad" states, in which the system is not working well or has failed.

The indicators used in reliability theory are the following:

(i) the *reliability function* R gives the probability that the system was always working from time 0 to time t:

$$R(t)=P\left(S_u\in U,\forall u\in[0,t]\right), \tag{12.35}$$

(ii) the *pointwise availability function* A gives the probability that the system is working at time t whatever happens on $(0,t)$:

$$A(t)=P\left(S_t\in U\right), \tag{12.36}$$

(iii) the *maintainability function* M gives the probability that the system, being in D on $[0,t)$, will leave set D at time t:

$$M(t)=P\left(S_u\in D, u\in[0,t), S_t\in U\right). \tag{12.37}$$

12.5. Markov renewal processes, semi-Markov and associated counting processes

Let us consider an SMC of kernel **Q**; we then have the following definitions.

Definition 12.7 *The two-dimensional process* $(J,T)=((J_n,T_n),n\geq 0)$ *where* T_n *is given by relation* (12.4) *is called a Markov renewal sequence or Markov renewal process.*

Cinlar (1969) also gives the term *Markov additive process*. It is justified by the fact that, using relation (12.5), we obtain:

$$\begin{aligned}&P(J_{n+1}=j,T_{n+1}\leq x\,|(J_k,T_k),k=0,\ldots,n)=\\&P(J_{n+1}=j,X_{n+1}\leq x-T_n\,|(J_k,T_k),k=0,\ldots,n)=\mathrm{Q}_{J_n j}(x-T_n).\end{aligned} \tag{12.38}$$

This last equality shows that the (J,T) process is a Markov process with $I\times\mathbb{R}^+$ as state space and having the "additive property":

$$T_{n+1}=T_n+X_{n+1}. \tag{12.39}$$

Let us state that according to the main definitions of Chapter 11, Definition 11.4, and always in the case of positive (J,X) chains, the random variables $T_n,(n\geq 0)$ are from now on called *Markov renewal times* or simply *renewal times*, the random variables $X_n,(n\geq 1)$ *interarrival* or *sojourn times* and the random variables $J_n,(n\geq 0)$ the *state variables*.

We will now define the *counting processes* associated with any Markov renewal process (MRP) as we did in the special case of renewal theory.

For any fixed time t, the r.v. $N(t)$ represents the *total number of jumps or transitions of the* (J,X) *process on* $(0,t]$, including possible transitions from any state towards itself (virtual transitions), assuming transitions are observable.

We have:

$$N(t)>t\Leftrightarrow T_n\leq t. \tag{12.40}$$

However here, we can be more precise and only count the total number of passages in a fixed state I always in $(0,t]$ represented by the r.v. $N_i(t)$.

Clearly, we can write:

$$N(t) = \sum_{i=1}^{m} N_i(t), t \geq 0 . \tag{12.41}$$

Definition 12.8 *With each Markov renewal process, the following* m+1 *stochastic processes are associated respectively with values in* $\mathbb{N}$ *:*

(i) the N-process ($N(t)$, $t \geq 0$);

(ii) the N_i-process ($N_i(t)$, $t \geq 0$), i=1,…,m,

respectively called the associated total counting process and the associated partial counting processes with of course:

$$N(0)=0,\ N_i(0)=0,\ i=1,\ldots,m. \tag{12.42}$$

It is now easy to introduce the notion of a *semi-Markov process* by considering at time t, the state entered at the last transition before or at t, that is, $J_{N(t)}$.

Definition 12.9 *With each Markov renewal process, we associate the following stochastic Z-process with values in I:*

$$Z=(Z(t),\ t \geq 0), \tag{12.43}$$

with:

$$Z(t)=J_{N(t)}. \tag{12.44}$$

This process will be called the associated semi-Markov process or simply the semi-Markov process (*SMP*) *of kernel* **Q**.

Remark 12.3

1) We will often use counting variables including the initial renewal, that is:

$$\begin{aligned} N'(t) &= N(t)+1, \\ N_i'(t) &= N_i(t)+\delta_{iJ_0}. \end{aligned} \tag{12.45}$$

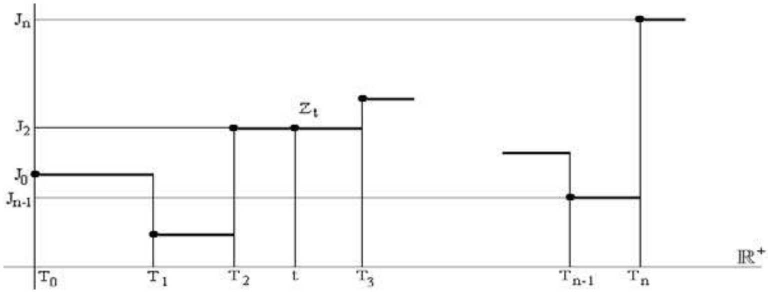

Figure 12.1. *A trajectory of an SMP*

2) Figure 12.1 gives a typical trajectory of MRP and SMP.

3) It is now clear that we can immediately consider an MRP defined by kernel **Q** without speaking explicitly of the basic (J,X) process with the same kernel **Q**, because the basic property (12.11) is equivalent to (12.38).

12.6. Particular cases of MRP

We will devote this section to particular cases of MRP having the advantage to lead to some explicit results.

12.6.1. *Renewal processes and Markov chains*

For the sake of completeness, let us first state that with $m = 1$, that is, that the observed system has only one possible state, the kernel **Q** has only one element, say the d.f. **F**, and the process $(X_n, n > 0)$ is a *renewal process*.

Secondly, to obtain *Markov chains* studied in Chapter 11, it suffices to choose for matrix **F** the following special degenerating case:

$$F_{ij} = U_1, \forall i, j \in I \tag{12.46}$$

and of course an arbitrary Markov matrix **P**.

This means that all r.v. X_n have a.s. the value 1, and so the single random component is the (J_n) process, which is, from relation (12.15), a homogenous MC of transition matrix **P**.

12.6.2. *MRP of zero order (Pyke (1962))*

There are two types of such processes.

12.6.2.1. *First type of zero order MRP*

This type is defined by the following semi-Markov kernel

$$\mathbf{Q} = \left[p_i F_i \right], \tag{12.47}$$

so that:

$$p_{ij} = p_i, F_{ij} = F_i, j \in I. \tag{12.48}$$

Naturally, we assume that for every i belonging to I, p_i is strictly positive.

In this present case, we discover that the r.vs. $J_n, n \geq 0$ are independent and identically distributed and moreover that the conditional interarrival distributions do not depend on the state to be reached, so that, by relation (12.22),

$$H_i = F_i, i \in I. \tag{12.49}$$

Moreover, since:

$$P(X_n \leq x \,|\, (J_k, X_k), k \leq n-1, J_n) = F_{J_{n-1}}(x), \tag{12.50}$$

we obtain:

$$P(X_n \leq x \,|\, (X_k), k \leq n-1) = \sum_{j=1}^{m} p_j F_j(x). \tag{12.51}$$

Introducing the d.f. F defined as

$$F = \sum_{j=1}^{m} p_j F_j, \tag{12.52}$$

the preceding equality shows that, for an MRP of zero order of the first type, the sequence $(X_n, n \geq 1)$ is a renewal process characterized by the d.f. F.

12.6.2.2. *Second type of zero order MRP*

This type is defined by the following semi-Markov kernel

$$\mathbf{Q} = \left[p_i F_j \right], \tag{12.53}$$

so that:

$$p_{ij} = p_i, F_{ij} = F_j, \; i, j \in I. \tag{12.54}$$

Here too, we suppose that for every i belonging to I, p_i is strictly positive.

Once again, the r.v. $J_n, n \geq 0$ are independent and equi-distributed and moreover the conditional interarrival distributions do not depend on the state to be *left*, so that, by relation (12.22)

$$H_i = \sum_{j=1}^{m} p_j F_j (= F), i \in I. \tag{12.55}$$

Moreover, since:

$$P(X_n \leq x \,|\, (J_k, X_k), k \leq n-1, J_n) = F_{J_n}(x), \tag{12.56}$$

we obtain

$$P(X_n \leq x \,|\, (X_k), k \leq n-1) = \sum_{j=1}^{m} p_j F_j(x) = F(x). \tag{12.57}$$

The preceding equality shows that, for an MRP of zero order of the second type, the sequence $(X_n, n \geq 1)$ is a renewal process characterized by the d.f. F as in the first type.

The basic reason for these similar results is that these two types of MRP are the *reverse*s (timewise) of each other.

12.6.3. *Continuous Markov processes*

These processes are defined by the following particular semi-Markov kernel

$$\mathbf{Q}(x) = \left[p_{ij} \left(1 - e^{-\lambda_i x} \right) \right], x \geq 0, \tag{12.58}$$

where $\mathbf{P} = \left[p_{ij} \right]$ is a stochastic matrix and where parameters $\lambda_i, i \in I$ are strictly positive.

The standard case corresponds to that in which $p_{ii} = 0, i \in I$ (see Chung (1960)). From relation (12.58), we obtain:

$$F_{ij}(x) = 1 - e^{-\lambda_i x}. \tag{12.59}$$

Thus, the d.f. of sojourn time in state *i* has an exponential distribution depending uniquely upon the occupied state *i*, such that both the excess and age processes also have the same distribution.

For $m = 1$, we obtain the usual Poisson process of parameter λ.

12.7. Markov renewal functions

Let us consider an MRP of kernel **Q** and to avoid trivialities, we will assume that:

$$\sup_{i,j} Q_{ij}(0) < 1, \tag{12.60}$$

where the functions Q_{ij} are defined by relation (12.7).

If the initial state J_0 is *i*, let us define the r.v. $T_n\left(i \middle| i\right), n \geq 1$, as the times (possibly infinite) of *successive returns* to state *i*, also called *successive entrance times* into $\{i\}$.

From the regenerative property of MRP, whenever the process enters into state *i*, say at time *t*, the evolution of the process on $\left[t, \infty\right)$ is probabilistically the same as if we had started at time 0 in the same state *i*.

It follows that the process $\left(T_n\left(i \middle| i\right), n \geq 0\right)$ with:

$$T_0\left(i \middle| i\right) = 0 \tag{12.61}$$

is a renewal process that could possibly be defective.

From now on, the r.v. $T_n\left(i \middle| i\right)$ will be called the *nth return time to state i*.

More generally, let us also fix state *j*, different from the state *i* already fixed; we can also define the *nth return* or *entrance time to state j*, but *starting from i as the initial state*. This time, possibly infinite as well, will be represented by $\left(T_n\left(j\middle|i\right), n \geq 0\right)$, also using the convention that

$$T_0\left(j\middle|i\right) = 0 . \tag{12.62}$$

Now, the sequence $\left(T_n\left(j\middle|i\right), n \geq 0\right)$ is a delayed renewal process with values in $\mathbb{R}^+$.

It is thus defined by two d.fs.: G_{ij} being that of $T_1\left(j\middle|i\right)$ and G_{jj} that of $T_2\left(j\middle|i\right) - T_1\left(j\middle|i\right)$, so that:

$$\begin{aligned} G_{ij}(t) &= P\left(T_1\left(j\middle|i\right) \leq t\right), \\ G_{jj}(t) &= P\left(T_n\left(j\middle|i\right) - T_{n-1}\left(j\middle|i\right) \leq t\right),\ n \geq 2. \end{aligned} \tag{12.63}$$

Of course, the d.f. G_{jj} suffices to define the renewal process $\left(T_n\left(j\middle|j\right),\ n \geq 0\right)$.

Remark 12.4 From the preceding definitions, we can also write that:

$$\begin{aligned} &G_{ij}(t) = P\left(N_j(t) > 0 \middle| J_0 = i\right);\ i, j \in I, \\ &P\left(T_1\left(j\middle|i\right) = +\infty\right) = 1 - G_{ij}(+\infty) \end{aligned} \tag{12.64}$$

and for the mean of the $T_n\left(i\middle|i\right), n \geq 1$, possibly infinite, we obtain:

$$\mu_{ij} = E\left(T_1\left(j\middle|i\right)\right) = \int_0^\infty t dG_{ij}(t), \tag{12.65}$$

with the usual convention that

$$0 \cdot (+\infty) = 0. \tag{12.66}$$

The means $\mu_{ij}, i, j \in I$ are called the *first entrance* or *average return times*.

It is possible to show that the functions $G_{ij}, i, j \in I$ satisfy the following relationships:

$$G_{ij}(t) = \sum_{k=1}^{m} G_{kj} \bullet Q_{ik}(t) + (1 - G_{jj}) \bullet Q_{ij}(t), i, j \in I, t \geq 0. \tag{12.67}$$

We will now define by A_{ij} and R_{ij} the associated *renewal functions*

$$A_{ij}(t) = E(N_j(t) \mid J_0 = i),$$
$$R_{ij}(t) = E(N'_j(t) \mid J_0 = i) \tag{12.68}$$

and by relations (12.45) we have:

$$R_{ij}(t) = \delta_{ij} U_0(t) + A_{ij}(t). \tag{12.69}$$

Using classical results of renewal theory (see Janssen and Manca (2006)) we obtain:

$$R_{jj}(t) = \sum_{n=0}^{\infty} G_{jj}^{(n)}(t),\ j \in I,$$
$$R_{ij}(t) = G_{ij} \bullet R_{jj}(t), \tag{12.70}$$

or equivalently, we have:

$$R_{ij}(t) = \delta_{ij} U_0(t) + G_{ij} \bullet \sum_{n=0}^{\infty} G_{jj}^{(n)}(t), i, j \in I. \tag{12.71}$$

Proposition 12.5 *Assumption* $m < \infty$ *implies that:*

(i) at least one of the renewal processes $\left(T_n\left(j \middle| j\right), n \geq 0\right), j \in I$ is not defective;

(ii) for all i belonging to I, there exists a state s such that

$$\lim_n T_n\left(s \middle| i\right) = +\infty, \text{ a.s.}; \tag{12.72}$$

(iii) for the r.v. T_n defined by relation (12.4), given that J_0=i whatever i is, we have a.s. that

$$\lim_n T_n = +\infty. \tag{12.73}$$

The following relations will express the renewal functions $R_{ij},\ i, j \in I$ in function of the kernel **Q** instead of the m^2 functions G_{ij}.

Proposition 12.6 *For every i and j of I, we have:*

$$R_{ij}(t) = \sum_{n=0}^{\infty} Q_{ij}^{(n)}(t). \tag{12.74}$$

Using matrix notation with:

$$\mathbf{R} = \left[R_{ij} \right], \tag{12.75}$$

relation (12.74) takes the form:

$$\mathbf{R} = \sum_{n=0}^{\infty} \mathbf{Q}^{(n)}. \tag{12.76}$$

Let us now introduce the *L-S transform of matrices.*

For any matrix of suitable functions A_{ij} from $\mathbb{R}^+$ to $\mathbb{R}$ represented by

$$\mathbf{A} = \left[A_{ij} \right] \tag{12.77}$$

we will represent its L-S transform by:

$$\overline{\mathbf{A}} = \left[\overline{A}_{ij} \right] \tag{12.78}$$

with

$$\overline{A}_{ij}(s) = \int_0^{\infty} e^{-st} dA_{ij}(t). \tag{12.79}$$

Doing so for matrix $\mathbf{R}$, we obtain the matrix form of relation (12.76),

$$\overline{\mathbf{R}}(s) = \sum_{n=0}^{\infty} \left(\overline{\mathbf{Q}}(s) \right)^n . \tag{12.80}$$

From this last relation, a simple algebraic argument shows that, for any s>0, relations

$$\overline{\mathbf{R}}(s)(\mathbf{I} - \overline{\mathbf{Q}}(s)) = (\mathbf{I} - \overline{\mathbf{Q}}(s)\overline{\mathbf{R}}(s) = \mathbf{I} \tag{12.81}$$

hold and so we also have:

$$\bar{\mathbf{R}}(s) = (\mathbf{I} - \bar{\mathbf{Q}}(s))^{-1}. \tag{12.82}$$

We have thus proved the following proposition.

Proposition 12.7 *The Markov renewal matrix* $\mathbf{R}$ *is given by*

$$\mathbf{R} = \sum_{n=0}^{\infty} \mathbf{Q}^{(n)}, \tag{12.83}$$

the series being convergent in $\mathbb{R}^+$*, and the inverse existing for all positive s.*

The knowledge of the Markov renewal matrix $\mathbf{R}$ or its L-S transform $\bar{\mathbf{R}}$ leads to useful expressions for d.f. of the first entrance times.

12.8. The Markov renewal equation

This section will extend the basic results related to the renewal equation developed in section 11.4 to the Markov renewal case.

Let us consider an MRP of kernel **Q**.

From relation (12.74), we obtain:

$$\begin{aligned} R_{ij}(t) &= \delta_{ij} U_0(t) + \sum_{n=1}^{\infty} Q_{ij}^{(n)}(t) \\ &= \delta_{ij} U_0(t) + \left(Q \bullet R\right)_{ij}(t). \end{aligned} \tag{12.84}$$

Using matrix notation with:

$$\mathbf{I}(t) = \left[\delta_{ij} U_0(t)\right], \tag{12.85}$$

relations (12.84) take the form:

$$\mathbf{R}(t) = \mathbf{I}(t) + \mathbf{Q} \bullet \mathbf{R}(t). \tag{12.86}$$

This integral matrix equation is called the *Markov renewal equation* for **R**.

To obtain the corresponding matrix integral equation for the matrix

$$\mathbf{H} = \left[H_{ij} \right], \tag{12.87}$$

we know, from relation (12.76), that

$$\mathbf{R}(t) = \mathbf{I}(t) + \mathbf{H}(t). \tag{12.88}$$

Inserting this expression of $\mathbf{R}(t)$ in relation (12.86), we obtain:

$$\mathbf{H}(t) = \mathbf{Q}(t) + \mathbf{Q} \bullet \mathbf{H}(t) \tag{12.89}$$

which is the *Markov renewal equation for* **H**.

For m=1, this last equation gives the traditional renewal equation.

In fact, the Markov renewal equation (12.86) is a particular case of the matrix integral equation of the type:

$$\mathbf{f} = \mathbf{g} + \mathbf{Q} \bullet \mathbf{f}, \tag{12.90}$$

called an integral equation of *Markov renewal type* (MRT), where

$$\mathbf{f} = \left(f_1,\dots, f_m \right)', \mathbf{g} = \left(g_1,\dots, g_m \right)' \tag{12.91}$$

are two column vectors of functions having all their components in B, the set of single-variable measurable functions, bounded on finite intervals, or to B^+ if all their components are non-negative.

Proposition 12.7 *The Markov integral equation of MRT,*

$$\mathbf{f} = \mathbf{g} + \mathbf{Q} \bullet \mathbf{f} \tag{12.92}$$

with **f**, **g** *belonging to* B^+*, has the unique solution:*

$$\mathbf{f} = \mathbf{R} \bullet \mathbf{g}. \tag{12.93}$$

12.9. Asymptotic behavior of an MRP

We will give asymptotic results, first for the Markov renewal functions and then for solutions to integral equations of an MRT. To conclude, we will apply these results to transition probabilities of an SMP.

We know that the renewal function R_{ij}, i, j belonging to I, is associated with the delayed renewal process, possibly transient, characterized by the couple (G_{ij},G_{jj}) d.f. on $\mathbb{R}^+$.

Let us recall that μ_{ij} represents the mean, possibly infinite, of the d.f. G_{ij}.

Proposition 12.8 *For all i, j of I, we have:*

$$\text{(i)} \lim_{t\to\infty} \frac{R_{ij}(t)}{t} = \frac{1}{\mu_{jj}}, \tag{12.94}$$

$$\text{(ii)} \lim_{t\to\infty} \frac{R_{ij}(t)-R_{ij}(t-\tau)}{\tau} = \frac{\tau}{\mu_{jj}}, \text{ for every fixed } \tau. \tag{12.95}$$

The next proposition, due to Barlow (1962), is a useful complement to the last proposition as it gives a method for computing the values of the mean return times $\mu_{jj}, j \in I$, in the ergodic case.

Proposition 12.9 *For an ergodic MRP, the mean return times satisfy the following linear system:*

$$\mu_{ij} = \sum_{k\neq j} p_{ik}\mu_{kj} + \eta_i, i = 1,\ldots,m. \tag{12.96}$$

In particular, for i=j, we have $\mu_{jj} = \dfrac{1}{\pi_j}\sum_k \pi_k\eta_k, \; j = 1,\ldots,m,$ (12.97)

where the $\eta_i, i \in I$ *are defined by relation* (12.25), *and where* $\boldsymbol{\pi} = (\pi_1,\ldots,\pi_m)$ *is the unique stationary distribution of the imbedded Markov chain.*

Remark 12.5 In a similar manner, Barlow (1962) proved that if $\mu_{ij}^{(2)}, i, j \in I$ is the second order moment related to the d.f. G_{ij}, then:

$$\mu_{ij}^{(2)} = \eta_i^{(2)} + \sum_{k \neq j} p_{ik} (\mu_{ik}^{(2)} + 2b_{ik} \mu_{kj}) \tag{12.98}$$

and in particular for $i = j$:

$$\mu_{jj}^{(2)} = \frac{1}{\pi_j} (\sum_k \pi_k \eta_k^{(2)} + 2 \sum_{k \neq j} \sum_l \pi_l p_{lk} b_k \mu_{kj}) \tag{12.99}$$

with

$$\eta_k^{(2)} = \int_{[0,\infty)} x^2 dH_k(x), k \in I, \tag{12.100}$$

provided that these quantities are finite.

12.10. Asymptotic behavior of SMP

12.10.1. *Irreducible case*

Let us consider the SMP ($Z(t)$, $t \geq 0$) associated with the MRP of kernel **Q** and defined by relation (12.43).

Starting with $Z(0) = i$, it is important for the applications to know the probability of being in state j at time t, that is:

$$\phi_{ij}(t) = P\left(Z(t) = j \middle| Z(0) = i\right). \tag{12.101}$$

A simple probabilistic argument using the regenerative property of the MRP gives the system satisfied by these probabilities as a function of the kernel **Q**

$$\phi_{ij}(t) = \delta_{ij}(1 - H_i(t)) + \sum_k \int_0^t \phi_{kj}(t-y) dQ_{ik}(y), \ i, j \in I. \tag{12.102}$$

It is also possible to express the transition probabilities of the SMP with the aid of the first passage time distributions $G_{ij}, i, j \in I$:

$$\phi_{ij}(t) = \phi_{jj} \bullet G_{ij}(t) + \delta_{ij}(1 - H_i(t)), \ i, j \in I. \tag{12.103}$$

If we fix the value j in relation (12.102), we see that the m relations for i=1,…,m form a Markov renewal type equation (MRE) of form (12.92).

Applying Proposition 12.7, we immediately obtain the following proposition.

Proposition 12.10 *The matrix of transition probabilities*

$$\mathbf{\Phi} = \left[\phi_{ij} \right] \tag{12.104}$$

is given by

$$\mathbf{\Phi} = \mathbf{R} \bullet (\mathbf{I} - \mathbf{H}) \tag{12.105}$$

with

$$\mathbf{H} = \left[\delta_{ij} H_i \right]. \tag{12.106}$$

So, instead of relation (12.103), we can now write:

$$\phi_{ij}(t) = \int_{[0,t]} (1 - H_j(t-y)) dR_{ij}(y). \tag{12.107}$$

Remark 12.6 *Probabilistic interpretation of relation* (12.107).

This interpretation is analogous to that of the renewal density given in Chapter 11.

Remark 12.7 The "infinitesimal" quantity $dR_{ij}(y)$ ($=r_{ij}(y)dy$, if $r_{ij}(y)$ is the density of function R_{ij}, if it exists) represents the probability that there is a Markov renewal into state j in the time interval $(y, y+dy)$, starting at time 0 in state i.

Of course, the factor $(1-H_j(t-y))$ represents the probability of not leaving state j before a time interval of length $t-y$.

The behavior of transition probabilities of matrix (12.104) will be given in the next proposition.

Proposition 12.11 *If* $Z = \left(Z(t), t \geq 0 \right)$ *is the SMP associated with an ergodic MRP of kernel* $\mathbf{Q}$*; then:*

$$\lim_{t \to \infty} \phi_{ij}(t) = \frac{\pi_j \eta_j}{\sum_k \pi_k \eta_k}, \; i, j \in I. \tag{12.108}$$

Remark 12.8

(i) As the limit in relation (12.108) does not depend on *i*, Proposition 12.11 establishes an *ergodic property* stating that:

$$\lim_{t\to\infty}\phi_{ij}(t)=\Pi_j,$$
$$\Pi_j=\frac{\pi_j\eta_j}{\sum_k \pi_k\eta_k}. \tag{12.109}$$

(ii) As the number *m* of states is finite, it is clear that $\left(\Pi_j, j\in I\right)$ is a probability distribution. Moreover, as $\pi_j > 0$ for all *j* (see relation (11.89)), we also have

$$\Pi_j > 0, j\in I. \tag{12.110}$$

So, asymptotically, every state is reachable with a strictly positive probability.

(iii) In general, we have:

$$\lim_{n\to\infty} p_{ij}^{(n)} \neq \lim_{t\to\infty}\phi_{ij}(t) \tag{12.111}$$

since of course

$$\pi_j \neq \Pi_j, j\in I. \tag{12.112}$$

This shows that the limiting probabilities for the embedded Markov chain are not, in general, the same as taking limiting probabilities for the SMP.

From Propositions 12.11 and 12.9, we immediately obtain the following corollary.

Corollary 12.1 *For an ergodic MRP, we have:*

$$\Pi_j=\frac{\pi_j}{\mu_{jj}}. \tag{12.113}$$

This result states that the limiting probability of being in state *j* for the SMP is the ratio of the mean sojourn time in state *j* to the mean return time of *j*.

This intuitive result also shows how the different return times and sojourn times have a crucial role in explaining why we have relation (12.113) as, indeed, for the imbedded MC, these times have no influence.

12.10.2. *Non-irreducible case*

It is often the case that stochastic models used for applications need non-irreducible MRP, as, for example, in the presence of an absorbing state, i.e. a state j such that

$$p_{jj} = 1. \tag{12.114}$$

We will now see that the asymptotic behavior is easily deduced from the irreducible case studied above.

12.10.2.1. *Uni-reducible case*

As for Markov chains, this is simply the case in which the imbedded MC is uni-reducible so that there exist l ($l<m$) transient states, and so that the other m-l states form a recurrent class C.

We always suppose aperiodicity both for the imbedded MC and for the considered MRP.

Let $T=\{1,...,l\}$ be the set of transient states $(T = I - C)$. From Corollary 11.2 we know that:

$$\lim_{t\to\infty} \phi_{ij}(t) = 0,\ \ i, j \in T. \tag{12.115}$$

Moreover, from Proposition 12.11 and relation (12.103):

$$\lim_{t\to\infty} \phi_{ij}(t) = G_{ij}(\infty)\frac{\pi_j \eta_j}{\sum_{k=l+1}^{m} \pi_k \eta_k},\ \ i, j \in C, \tag{12.116}$$

where $(\pi_{l+1},...,\pi_m)$ represents the unique stationary probability distribution of the sub-Markov chain with C as state space.

Since

$$G_{ij}(\infty) = f_{ij}, \tag{12.117}$$

we obtain

$$G_{ij}(\infty) = f_{i,C}, \tag{12.118}$$

where $f_{i,C}$ is the probability that the system, starting in state i will be absorbing by the recurrent class C.

As there is only one essential class, we know that for all states i of I:

$$f_{i,C}=1, \tag{12.119}$$

thus proving the following proposition.

Proposition 12.12 *For any periodic uni-reducible MRP, we have:*

$$\lim_{t\to\infty}\phi_{ij}(t)=\Pi'_j,\ j\in I, \tag{12.120}$$

where

$$\Pi'_j=\begin{cases}0, & j\in T,\\ \dfrac{\pi_j\eta_j}{\sum_{k=l+1}^{m}\pi_k\eta_k}, & j\in C.\end{cases} \tag{12.121}$$

Here too, as the limit in (12.121) is independent of the initial state i, this result gives an *ergodic property*.

12.10.2.2. *General case*

For any aperiodical MRP, there exists a unique partition of the state space I:

$$I=T\cup C_1\cup\cdots\cup C_r,\ \ r<m, \tag{12.122}$$

where T represents the set of transient states and $C_\nu,\nu=1,\dots,r$ represents the ν th essential class necessarily formed of positive recurrent.

From Chapter 11, we know that the system will finally enter one of the essential classes and will then stay in it forever. Thus, a slight modification of the last proposition leads to the next result.

Proposition 12.13 *For any aperiodic MRP, we have:*

$$\lim_{t\to\infty}\phi_{ij}(t)=\Pi'_{ij},i,j\in I, \tag{12.123}$$

with, for any $j\in C_\nu,\nu=1,\dots,r$ *:*

$$\Pi'_{ij} = \begin{cases} \Pi'^{\nu}_{j}, & i \in C_{\nu}, \nu = 1,\ldots,r, \\ 0, & i \in C_{\nu'}, \nu \neq \nu', \nu' = 1,\ldots,r, \\ f_{i,C_{\nu}} \Pi'^{\nu}_{j}, & i \in T \end{cases} \tag{12.124}$$

where $(\Pi'^{\nu}_{j}, j \in C_{\nu})$ *is the only stationary distribution of the sub-SMP with* C_{ν} *as state space, that is:*

$$\Pi'_{j} = \frac{\pi_j \eta_j}{\sum_{k \in C_{\nu}} \pi_k \eta_k}, \tag{12.125}$$

where $\left(\pi^{\nu}_{k}, k \in C_{\nu}\right)$ *is the unique stationary distribution of the sub-Markov chain with* C_{ν} *as the state space and* $\left(f_{i,C_{\nu}}, i \in T\right)$ *is the unique solution of the linear system*

$$y_i - \sum_{j \in T} p_{ij} y_j = \sum_{j \in C_{\nu}} p_{ij}, i \in T. \tag{12.126}$$

Note that, in this proposition, the ergodic property is lost; this is due to the presence of the quantities $f_{i,C_{\nu}}$ in relation (12.124).

12.11. Non-homogenous Markov and semi-Markov processes

To finish this chapter, let us recall the basic definitions and results for the *non-homogenous* case for which time itself has influence on the transition probabilities. Due to the importance of its applications, in particular within insurance, we carefully develop some special cases such as non-homogenous Markov processes.

12.11.1. *General definitions*

To begin, we present the general definition of non-homogenous semi-Markov processes (NHSMP) including, as particular cases, non-homogenous Markov processes (NHMP) in continuous time, non-homogenous Markov chains (NHMC) in discrete time and non-homogenous renewal processes (NHRP).

We follow the original presentation given by Janssen and De Dominicis (1984).

12.11.1.1. *Completely non-homogenous semi-Markov processes*

As usual, let us consider a system *S* having *m* possible states constituting the set $I=\{1,\ldots,m\}$ defined on the probability space $\left(\Omega,\Im,P\right)$.

Definition 12.10 *The two-dimensional process in discrete time* $\left((J_n,X_n),n\geq 0\right)$ *with values in* $I\times\mathbb{R}^+$ *such that:*

$$\begin{aligned}&J_0=i,X_0=0,a.s.,i\in I,\\&P(J_n=j,X_n\leq x\,|\,(J_k,X_k),k\leq n-1)={}^{(n-1)}Q_{J_{n-1}j}(T_{n-1},T_{n-1}+x),\\&j\in I,x\in\mathbb{R}^+,\\&T_0=0,T_n=\sum_{k=0}^{n}X_k,a.s.\end{aligned}\tag{12.127}$$

is called a completely non-homogenous semi-Markov chain (*CNHSMC*) *of kernel* $\mathbf{Q}(s,t)=\left({}^{(n-1)}\mathbf{Q}(s,t),n\geq 1\right)$.

Consequently, the past influences the evolution of the process by the presence of T_{n-1} and *n* in (12.127).

Definition 12.11 *The sequence* $\mathbf{Q}=\left({}^{(n-1)}\mathbf{Q}(s,t),n\geq 1\right)$ *of* $m\times m$ *matrices of measurable functions of* $\mathbb{N}_0\times\mathbb{R}^+\times\mathbb{R}^+\mapsto\left[0,1\right]$ *where*:

$${}^{(n-1)}\mathbf{Q}(s,t)=\left[{}^{(n-1)}Q_{ij}(s,t)\right]\tag{12.128}$$

and satisfying the following conditions:

$$\begin{aligned}&\text{(i) } \forall n>0,\forall i,j\in I,\forall t,s\in\mathbb{R}^+:t\leq s\Rightarrow {}^{(n-1)}Q_{ij}(s,t)=0,\\&\text{(ii) } \forall n>0,\forall i\in I,\forall s\in\mathbb{R}^+:\sum_{j=1}^{n}{}^{(n-1)}Q_{ij}(s,\infty)=1,\\&\text{with } {}^{(n-1)}Q_{ij}(s,\infty)=\lim_{t\to\infty}{}^{(n-1)}Q_{ij}(s,t),\end{aligned}\tag{12.129}$$

is called a completely non-homogenous semi-Markov (*CNHSM*) *kernel*.

Clearly, for all fixed *s*, ${}^{(n-1)}Q_{ij}(s,.)$ is a mass function, zero for $t\leq s$.

Definition 12.12 *For all* $i, j \in I, n \in \mathbb{N}_0, s,t \in \mathbb{R}^+$, *functions* ${}^{(n-1)}p_{ij}(s)$, ${}^{(n-1)}H_{ij}(s,t)$, ${}^{(n-1)}F_{ij}(s,t)$ *are defined as follows:*

$$
\begin{aligned}
&{}^{(n-1)}p_{ij}(s) = {}^{(n-1)}Q_{ij}(s,\infty),\\
&{}^{(n-1)}H_{ij}(s,t) = \sum_j {}^{(n-1)}Q_{ij}(s,t),\\
&{}^{(n-1)}F_{ij}(s,t) = \begin{cases} U_1(s)U_1(t), & {}^{(n-1)}p_{ij}(s) = 0,\\ \dfrac{{}^{(n-1)}Q_{ij}(s,t)}{{}^{(n-1)}p_{ij}(s)}, & {}^{(n-1)}p_{ij}(s) > 0. \end{cases}
\end{aligned}
\tag{12.130}
$$

Working as in section 12.3, it is easy to prove that we still have the following probabilistic meaning:

$$
\begin{aligned}
&{}^{(n-1)}p_{ij}(s) = P(J_n = j \mid J_{n-1} = i, T_{n-1} = s),\\
&{}^{(n-1)}H_i(s,t) = P(X_n \le t-s \mid J_{n-1} = i, T_{n-1} = s)(= P(T_n \le t \mid J_{n-1} = i, T_{n-1} = s)),\\
&{}^{(n-1)}F_{ij}(s,t) = P(X_n \le t-s \mid J_{n-1} = i, J_n = j, T_{n-1} = s)\\
&\qquad (= P(T_n \le t \mid J_{n-1} = i, J_n = j, T_{n-1} = s)).
\end{aligned}
\tag{12.131}
$$

In matrix notation, using the element by element product (Scott product) defined as:

$$
\begin{aligned}
&\mathbf{A} \cdot \mathbf{B} = \left[a_{ij} b_{ij} \right],\\
&\mathbf{A} = \left[a_{ij} \right], \mathbf{B} = \left[b_{ij} \right],
\end{aligned}
\tag{12.132}
$$

we will write:

$$
\begin{aligned}
&{}^{(n-1)}\mathbf{F}(s,t) = \left[{}^{(n-1)}F_{ij}(s,t) \right],\\
&{}^{(n-1)}\mathbf{P}(s) = \left[{}^{(n-1)}p_{ij}(s) \right],\\
&{}^{(n-1)}\mathbf{Q}(s,t) = {}^{(n-1)}\mathbf{P}(s) \cdot {}^{(n-1)}\mathbf{F}(s,t).
\end{aligned}
\tag{12.133}
$$

We can now give the following definitions similar to the traditional or homogenous semi-Markov theory presented in section 12.5.

Definition 12.13 *The counting process* $(N(t), t \geq 0)$ *defined as*

$$N(t) = \sup_n \{n : T_n \leq t\} \tag{12.134}$$

is called the associated counting process with the CNHSM kernel **Q**.

Definition 12.14 *The process* $\left((J_n, T_n), n \geq 0\right)$ *is called a completely non-homogenous Markov additive process or Markov renewal process* (*CNHMAP or CNHMRP*).

Definition 12.15 *The process* $Z = (Z(t), t \geq 0)$ *defined as*

$$Z(t) = \begin{cases} J_{N(t)}, N(t) < \infty, \\ \theta, N(t) < \infty, \end{cases} \tag{12.135}$$

where θ *is a new state added to I, is called the completely non-homogenous semi-Markov process* (*CNHSMP*) *of kernel* **Q**.

Definition 12.16 *The random variable L defined as*

$$L = \inf \{t : Z(t) = \theta\} \tag{12.136}$$

is called the lifetime of the CNHSMP Z.

Definition 12.17 *The associated counting process* $(N(t), t \geq 0)$ *or the CNHSMP Z=* $(Z(t), t \geq 0)$ *of kernel* **Q** *is explosive if and only if*

$$L = \infty, a.s. \tag{12.137}$$

and non-explosive if and only if

$$L < \infty, a.s. \tag{12.138}$$

For very general counting processes, De Vylder and Haezendonck (1980) have given necessary and sufficient conditions for non-explosion. Here, in general, we always assume non-explosive processes.

For the two-dimensional process $((J_n, T_n), n \geq 0)$, we have the following result:

$$\begin{aligned} &P\left(J_1 = j, T_1 \le t \middle| J_0 = i, T_0 = 0\right) = {}^{(0)}Q_{ij}(0,t)(= Q_{ij}^{(1)}(t)), \\ &P\left(J_2 = j, T_2 \le t \middle| J_0 = i, T_0 = 0\right) = \sum_k \int_0^t {}^{(1)}Q_{kj}(x,t)\, {}^{(0)}Q_{ij}(0,dx)(= Q_{ij}^{(2)}(t)), \end{aligned} \tag{12.139}$$

and in general

$$\begin{aligned} P\left(J_n = j, T_n \le t \middle| J_0 = i, T_0 = 0\right) &= \sum_k \int_0^t {}^{(n-1)}Q_{kj}(x,t)\, {}^{(1)}Q_{ij}(dx) \\ &= (Q_{ij}^{(n)}(t)), n > 1). \end{aligned} \tag{12.140}$$

Using matrix notation, we may write for two $m \times m$ matrices of mass functions $\mathbf{A}(t)$, $\mathbf{B}(t)$:

$$\int_0^t \mathbf{A}(t) d\mathbf{B}(t) = \left[\sum_{k=1}^n \int_0^t B_{kj}(z) dA_{ik}(z) \right], \tag{12.141}$$

and so relations (12.140) can be written in matrix form:

$$\begin{aligned} &\mathbf{Q}^{(n)}(t) = \int_0^t \mathbf{Q}^{(n-1)}(z,t) d\mathbf{Q}^{(1)}(x), n > 1, \\ &\text{with} \\ &\mathbf{Q}^{(n)}(t) = \left[Q_{ij}{}^{(n)}(t) \right], n \ge 1, \\ &\mathbf{Q}^{(1)}(t) = \left[{}^{(0)}Q_{ij}(0,t) \right]. \end{aligned} \tag{12.142}$$

In the particular class of traditional SMP, relation (12.142) gives the *n*-fold convolution of the SM kernel **Q**.

Another very important distribution is the marginal distribution of the *Z* process as it gives the state occupied by the system *S* at time *t*.

Let us introduce the following probabilities:

$${}^{(n)}\phi_{ij}(s,t) = P\left(Z(t) = j \middle| Z(0) = i, N(s-) < N(s), N(s) = n\right), i, j \in I, n \ge 0. \tag{12.143}$$

The conditioning means that $T_n = s$ and that there exists a transition at time *s* such that the new state occupied after the transition is *i*.

Clearly, these probabilities satisfy the following relations:

$$ {}^{(n)}\phi_{ij}(s,t) = \delta_{ij}(1 - {}^{(n)}H_i(s,t)) + \sum_{k\in I}\int_s^t {}^{(n)}\phi_{kj}(u,t)\,{}^{(n-1)}Q_{ik}(s,du), i,j \in I. \qquad (12.144) $$

From relation (12.127), it is clear that we have:

$$ P(J_n = j \,|\, (J_k, T_k), k \le n-1) = {}^{(n-1)}p_{J_{n-1}j}(T_{n-1}), a.s. \qquad (12.145) $$

It follows that the process $(J_n, n \ge 0)$ can be viewed as a *conditional multiple Markov chain*; this means that, given the sequence $(T_n, n \ge 0)$, each transition from $J_{n-1} \to J_n$ obeys a Markov chain of kernel ${}^{(n-1)}P(T_{n-1})$.

Definition 12.18 *The conditional multiple Markov chain* $(J_n, n \ge 0)$ *is called the embedded multiple MC.*

12.11.1.2. *Special cases*

Let us point out that Definition 12.18 is quite general as indeed it is non-homogenous both for the time *s* and for the number of transitions *n*, this last one giving the possibility to model *epidemiological* phenomena such as AIDS (see in Janssen and Manca (2006) the example of *Polya processes* and semi-Markov extensions).

This extreme generality gives importance to the following particular cases.

(i) Non-homogenous Markov additive process and semi-Markov process

Writing $\mathbf{Q}=\left({}^{(n-1)}\mathbf{Q}(s,t), n \ge 1\right)$, we have as first special case:

$$ {}^{(n-1)}\mathbf{Q}(s,t) = \mathbf{Q}(s,t), n \ge 1, s < t, \qquad (12.146) $$

that is $\mathbf{Q}$ independent of *n*, then the kernel $\mathbf{Q}$ is called a non-homogenous semi-Markov kernel (NHSMK) defining a non-homogenous Markov additive process (NHMAP) $((J_n, T_n), n \ge 0)$ and a non-homogenous semi-Markov process (NHSMP) $Z = (Z(t), t \ge 0)$.

This family was introduced in a different way by Hoem (1972).

It is clear that relation (12.146) means that the sequences

$$^{(n-1)}\mathbf{F}(s,t)=\mathbf{F}(s,t),\ ^{(n-1)}\mathbf{P}(s)=\mathbf{P}(s) \quad \forall n\geq 0 \tag{12.147}$$

are independent of n or equivalently that

$$\mathbf{Q}(s,t)=\mathbf{P}(s)\cdot\mathbf{F}(s,t). \tag{12.148}$$

Let us point out that, in this case, relation (12.144) becomes:

$$\phi_{ij}(s,t)=\delta_{ij}(1-H_i(s,t))+\sum_k\int_s^t\phi_{ij}(u,t)Q_{ik}(s,du), i,j\in I. \tag{12.149}$$

If moreover, we have

$$\mathbf{P}(s)=\mathbf{P}, s\geq 0, \tag{12.150}$$

then the kernel **Q** is called a *partially non-homogenous semi-Markov kernel* (PNHSMK) defining a *partially non-homogenous Markov additive process* (PNHMAP) $\left((J_n,T_n),n\geq 0\right)$ and a partially non-homogenous semi-Markov process (PNHSMP) $Z=(Z(t),t\geq 0)$.

This family was introduced in a different way by Hoem (1972).

(ii) Non-homogenous MC

If the sequences $^{(n-1)}\mathbf{P}(s),\forall s\geq 0$ are independent of s, then $\left(J_n,n\geq 0\right)$ is a traditional NHMC.

Homogenous Markov additive process

A PNHSMK **Q** such that

$$F(s,t)=F(t-s), s,t\geq 0, t-s\geq 0, \tag{12.151}$$

is of course a traditional homogenous SM kernel as in section 12.2.

(iii) Non-homogenous renewal process

For m=1, the CNHMRP of kernel **Q** is given by

$$\mathbf{Q}(s,t)=(^{(n-1)}\mathbf{F}(s,t)), s,t>0, t-s\geq 0 \tag{12.152}$$

and characterizes the sequence $(X_n,n\geq 0)$ with, as in (12.127),

$$\begin{aligned}&X_0 = 0, a.s.,\\&P(X_n \le x \mid X_k, k \le n-1) = {}^{(n-1)}F(T_{n-1}, T_{n-1} + x), x \in \mathbb{R}^+,\\&T_0 = 0, T_n = \sum_{k=0}^{n} X_k, a.s.\end{aligned} \tag{12.153}$$

In this case, the process $(X_n, n \ge 0)$ is called a *completely non-homogenous dependent renewal process* (CNHDRP) of kernel **Q**.

If, moreover,

$$^{(n-1)}F(s,t) = {}^{(n-1)}F(t-s), s,t > 0, t-s \ge 0, n \ge 1, \tag{12.154}$$

it follows that

$$\begin{aligned}&X_0 = 0, a.s.,\\&P(X_n \le x \mid X_k, k \le n-1) = {}^{(n-1)}F(x), x \in \mathbb{R}^+, n \ge 1\end{aligned} \tag{12.155}$$

and so the process $(X_n, n \ge 0)$ is a sequence of independent *t* r.v. called a *completely non-homogenous renewal process* (CNHRP) of kernel *F*.

Remark 12.9 In the non-homogenous case, it is much more difficult to obtain asymptotic results (see, for example, Benevento (1986) and Thorisson (1986) for interesting theoretical results). That is not so dramatic as we can say that non-homogenous models are used for modeling *transient* situations and not *asymptotic* ones, and that is why we personally think that all attention must be given to the construction of numerical methods, for example, to be able to solve the non-homogenous integral equations system (12.149) (see Janssen and Manca (2007)).

However, let us mention that, for the particular case of non-homogenous Markov chains, there exist more asymptotic results (see, for example, Isaacson and Madsen (1976)).